ISSN 2664-4150 (Print) & ISSN 2664-794X (Online)
**South Asian Research Journal of Engineering and Technology**
*Abbreviated Key Title: South Asian Res J Eng Tech*

**| Volume-5 | Issue-5 | Sep-Oct- 2023 |** **DOI: 10.36346/sarjet.2023.v05i05.003**

**Review Article**

# An Overview of Strategies for Conceptualizing Derivative and their Applications in Daily Life for Secondary-Level Mathematics Students

Rajendra Kunwar[1], Laxmi G. C[1*]
[1]Department of Mathematics Education, Tribhuvan University, Sanothimi Campus, Bhaktapur, Nepal

***Corresponding Author:** Laxmi G. C
Department of Mathematics Education, Tribhuvan University, Sanothimi Campus, Bhaktapur, Nepal

**Article History**
Received: 28.08.2023
Accepted: 03.10.2023
Published: 06.10.2023

**Abstract:** Derivative is a foundational concept in calculus but can be challenging for secondary students to grasp conceptually. This paper provides an overview of research-backed strategies for developing a strong conceptual understanding of derivative. Visual and intuitive approaches are discussed, such as relating the derivative to real-world contexts involving speed, growth, and modeling phenomena. The importance of multiple linked representations and building connections between graphical, numeric, verbal and symbolic perspectives is emphasized. Strategies for motivating learning through real-world applications and simulations relating concepts to students' lives are outlined. Procedural expertise and conceptual mastery are cultivated together. The benefits of a conceptual learning, problem solving abilities, and STEM-related fields are noted. Curricular recommendations focus on conceptual exploration prior to formal definitions. Thus, the paper highlights best practices for conceptualizing derivative through visual-intuitive, multi-representational and application-based approaches to promote flexible, adaptive understanding and lay the groundwork for calculus success.

**Keywords:** Calculus, Conceptual understanding, Derivatives, Rates of change, Visualization.

## INTRODUCTION

Derivative is the fundamental concept in calculus that builds upon the understanding of rates of change (Thompson, 1994). However, research has shown that many secondary-level students struggle to develop a conceptual understanding of derivatives due to their abstract and counterintuitive nature (Orton, 1983; Ferrini-Mundy & Graham, 1994). Without a coherent conceptual understanding, students often resort to rote memorization of procedures instead of the flexible application of derivative principles (Rasmussen, 2001). Extensive literature in mathematics education has focused on the concept of derivatives and its applications in various fields. Prominent calculus textbooks, such as Stewart (2015), Anton *et al*., (2012), and Larson *et al.*, (2013), highlight the fundamental role of derivatives and their relevance in real-life scenarios.

The definition of derivative as the rate at which a quantity changes with respect to another variable is a fundamental concept taught in secondary-level mathematics education (Stewart, 2015; Anton *et al*., 2012). The unique property of derivatives in capturing instantaneous rates of change and examining the behavior of functions at specific points is a well-established aspect of calculus (Stewart, 2015). To facilitate conceptual understanding, the geometric interpretation of derivatives is commonly employed. This approach relates derivative to the slopes of curves, is documented in various calculus textbooks (Stewart, 2015; Anton *et al*., 2012). By visualizing derivative in terms of geometric representations, students can better grasp the concept and its significance (Stewart, 2015). It is crucial for students to understand the distinction between average rate of change and instantaneous rate of change, which is emphasized in calculus textbooks (Stewart, 2015; Anton *et al*., 2012). Exploring the difference between these two concepts enables students to develop a deeper understanding of derivatives as tools for analyzing dynamic systems. The role of limits in derivative calculations is fundamental in calculus (Stewart, 2015; Anton *et al*., 2012). Taking the limit of a difference quotient is a standard procedure for calculating derivative. The notion of approaching infinitesimally small intervals, central to the concept of limits, is essential for understanding the mathematical foundation of derivative (Stewart, 2015).

The practical applications of derivative in daily life are well-documented in calculus. Stewart (2015) and Anton *et al*., (2012) emphasize the extensive use of derivative in the fields such as physics, economics, engineering, and computer science. These applications enable us to model and analyze real-world phenomena, ranging from the motion of objects to optimization problems. Secondary-level mathematics students can acquire a robust comprehension of derivative and enhance their problem-solving abilities with real-world application by implementing techniques to conceptualize derivative and exploring their practical applications in everyday life. The importance of these strategies for mathematics education has been emphasized by various authors and researchers in the field of calculus (Stewart, 2015; Anton *et al.*, 2012; Larson *et al*., 2013).

A strong conceptual understanding of derivative is crucial, as calculus lays the groundwork for more advanced math, science, and engineering courses (Drijvers *et al.*, 2016). Wewe (2020) argue that conceptual mastery in introductory calculus leads to better preparation for university-level mathematics. Furthermore, Huang (2011) found a significant link between conceptual proficiency in derivatives and achievement in forthcoming physics applications of calculus. The conceptual understanding of derivatives, which are fundamental concepts in calculus, frequently poses challenges for students in their educational journey (Orton, 1983; Drijvers *et al*., 2016). The abstract nature of derivatives can stance challenges for students unless presented with real-world contexts and practical applications (Brown, 2019). Research has demonstrated that when students are provided with strategies to conceptualize derivatives, it leads to improved educational outcomes (NCTM, 2014).

Derivatives have numerous practical applications in modeling dynamical systems and optimizing outcomes in various domains. Connecting derivatives to real-world problems motivates the learners and retention of abstract mathematical concepts, demonstrating the utility of calculus skills (Hyer & Gardner, 2007). The application of derivative in analyzing rates of change in natural phenomena fosters an appreciation for mathematics as a functional tool, supporting STEM interest development (Maltese & Tai, 2010). Mastery of derivative enables a comprehensive understanding of quantitative aspects in fields such as economics, medicine, and engineering, opening pathways to technical careers that shape modern life (National Research Council, 2012). Representing real situations mathematically through derivative clarifies conceptual foundations and builds quantitative reasoning, providing transferable skills applicable to diverse endeavors (Chauvot, 2009).

Understanding derivative also plays a crucial role in addressing complex societal issues. The ability to apply derivative principles aids in problem-solving, resource management, climate change modeling, trends of epidemiology, and many other areas where quantitative analysis is essential (National Research Council, 2012). By understanding derivative, individuals can engage in informed discussions and make decisions based on data-driven insights. Furthermore, a conceptual understanding of derivative promotes critical thinking and logical reasoning, empowering individuals to evaluate claims and arguments that involve rates of change and optimization.

In conclusion, derivatives are a fundamental concept in calculus with broad applications in various fields. However, students often struggle to develop a conceptual understanding of derivatives due to their abstract nature. Geometric interpretations, the distinction between average and instantaneous rates of change, and the role of limits are key components in facilitating conceptual understanding. Moreover, connecting derivative to real-world scenarios and applications enhances motivation and supports the development of problem-solving skills. A strong conceptual understanding of derivatives is essential for success in advanced mathematics and for applying mathematical principles in practical situations. It also equips individuals with transferable skills that are valuable in a wide range of domains and societal contexts.

### *Historical Context of Evolving Derivative*

The foundations of derivative can be traced back to ancient Indian, Chinese, Greek and Islamic mathematicians who explored infinitesimals and rate of change (Katz, 1998). Ancient mathematicians initially explored the concept of derivative to understand rates of change, although it was not until the 19th century, through the contributions of Cauchy and Weierstrass, that formal definitions incorporating limits were developed (Boyer, 1968). However, modern calculus originated in the 17th century with the independent work of Newton and Leibniz, who formalized derivative and integrals using the limit process (Boyer, 1968). Newton's seminal work "Methodus Fluxionum et Serierum Infinitarum" published in 1671, and introduced a graphical approach to conceptualizing derivative that represented a significant advancement at the time. He used physical notions of motion and a quantity's "fluxion" or instantaneous rate of change to geometrically link a varying function with the increments of its independent variable as it changed (White, 1961). Newton's approach visualized the derivative as the slope of the tangent line to a function's curve at a given point, representing the instantaneous velocity of the dependent variable. This built upon earlier work utilizing average rate of change over intervals, moving calculus toward the foundation of the limit that would later be formalized. Newton used physical analogies like motion to explain derivative in a more intuitive and visual way. His graphical interpretation, which involved diagrams and pictures, helped us understand derivative beyond just average rates. This approach was a pioneering effort in conceptualizing derivatives (White, 1961). Meanwhile, Leibniz published De Geometria surva, investigating a geometric interpretation of the derivative through infinitesimals (Leibniz, 1676). However, their notation and limit-based theories were not fully developed until later work by mathematicians such as Euler, Cauchy and Weierstrass in the 18th-19th centuries (Edwards, 1979).

Early instruction of calculus concepts took a procedural approach rooted in symbolic manipulation (Ferrini-Mundy & Graham, 1994). In the late 1900s, prior to the introduction of limits, early attempts to conceptualize derivative was made through geometric and intuitive approaches. These methods based on infinitesimals aimed to develop understandings of derivative before the concept of limits, but sometimes mixed up graphical and symbolic aspects (Tall, 2013). Through the 20th century, graphical approaches emphasizing visual reasoning for derivative emerged (Cornu, 1991). More recently, conceptual instruction stresses multiple representations, real-world connections and active-learning experiences to build robust understanding (Orton, 1983; White, 1961).

Over time, the conceptualization of derivative has progressed from ancient geometric intuitions to contemporary formalizations based on limits. Ongoing advancements in instructional strategies continue to incorporate visual-spatial, tangible, and applied perspectives (Atit *et al*., 2020). This reflects ongoing efforts to overcome epistemological obstacles and effectively impart both conceptual understanding and technical proficiency with this core calculus concept.

Ancient Greek mathematicians including Eudoxus, Archimedes and Apollonius made early contributions to conceptualizing rates of change and derivatives through investigations of exhaustion proofs and theorems involving tangents to curves (Netz, 2022). This built on earlier work by Indian mathematicians who studied infinitesimals in the spirit of calculus (Stillwell, 2010). In Asia, Chinese mathematician Liu Hui explored derivatives and rates of change through diagrams and algebraic formulas as early as the 3rd century (Cullen, 1996). Other Asian mathematicians such as Bhaskara II expanded this work, introducing basic concepts of maxima, minima and rates of change (Joseph, 2000). In the 17th century, Fermat introduced adequality and tangents to problems involving maxima and minima, influencing early conceptualizations of differentiation (Edwards, 1979). Cavalieri introduced a precursor to limits called method of indivisibles for finding lengths, areas, and tangents (Bell, 2008).

In the 20th century, research further expanded the conceptual foundations of derivative. Schwartz reworked analysis using distributions and generalized functions, connecting derivative to impulse type concepts (Schwartz, 1950). Tall incorporated cognitive science and proposed three worlds of mathematics - embodied, proceptual, and formal/axiomatic representations (Tall, 2013). Recent work focuses on multi-representational sense making through graphic, numeric, algebraic and contextual understandings. Technological tools and simulated environments also support dynamic exploration and conceptualizing derivative (Yang & Baldwin, 2020). This provides historical context on evolving perspectives of derivative from antiquity to present day instructional paradigms.

**Rationale of the Study**

Derivative, a fundamental concept in secondary-level calculus courses, often pose conceptual challenges for students (Orton, 1983; Sofronas *et al*., 2011). This study aims to address the issue by exploring effective conceptual teaching strategies, such as providing real-world contexts, which have been shown to enhance student understanding, motivation, and learning outcomes. Approaches that incorporate visual, tangible, and problem-solving elements align with research on meaningful learning and cater to diverse cognitive styles (Drijvers et al., 2016; NCTM, 2000). These strategies not only establish a strong foundation for further calculus topics (Huang, 2011) but also foster applied mathematical thinking, essential for developing 21st-century skills and preparing students for STEM fields (NRC, 2012). Furthermore, the insights gained from this topic can inform secondary-level teacher training programs and the development of instructional materials that promote conceptual understanding (Confrey & Lachance, 2000; Chauvot, 2009). The strategies discussed may also enhance comprehension of other challenging mathematical concepts (Lobato *et al*., 2012). Additionally, early exposure to applied perspectives can potentially increase long-term participation in STEM fields (Maltese & Tai, 2010). This study focus on addressing the recommendations of researchers regarding conceptual teaching that contributes to improved learning outcomes and holds significant relevance (NCTM, 2014).

The strategies discussed align with pedagogical frameworks such as embodied cognition and situated learning theories (Lakoff & Nunez, 2000). From a practical perspective, offering a comprehensive understanding of effective conceptual approaches for derivative holds great significance for multiple reasons. Firstly, it can inform secondary-level teacher education programs regarding constructivist teaching methods that are supported by research (Confrey & Lachance, 2000) that plays a crucial role in developing foundational understandings and applying conceptual approaches in teaching (Brown, 2019). Secondly, the strategies and examples discussed have implications for designing curriculum materials that align with standards and develop both conceptual and procedural fluency with derivative (NCTM, 2014). Such resources are important for classroom instruction. Thirdly, enhancing conceptualization may help reduce failure rates in calculus courses by strengthening students' knowledge bases, thereby improving learning outcomes. Lastly, a deeper and earlier comprehension of derivative could better prepare students for subsequent STEM courses that rely on a solid understanding of calculus foundations (Huang, 2011).

Finally, adopting an applied view of mathematics may help change mindsets and increase long-term participation in STEM fields by demonstrating the relevance of mathematical concepts (Maltese & Tai, 2010). This study contributes to the goals of improving STEM education. In a broader sense, tackling challenges by employing conceptually-grounded theoretical frameworks and empirically-supported strategies holds immense importance in mathematics education. It not only enhances learning outcomes but also contributes to students' future success across various disciplines.

**Objectives of the Study**

The objectives of the study are as follows.

(i) To conceptualize the concept and meaning of derivative.
(ii) To explore the different strategies for conceptualizing derivative.
(iii) To identify real-life applications of derivative and encourage to apply in their daily life situation.

# METHODOLOGY

This study utilizes a review-based research design to conceptualize the meaning and applications of derivative. The overarching goal is to synthesize approaches that enhance student understanding of derivative and their practical applications in daily life. The design focuses on conceptual change through multiple representations, real-world modeling, and technology-enhanced learning approaches. It examines the best ways to develop conceptual understanding of derivative to the learner exploring the relationship between intuitive mathematical knowledge, practical applications, and real life applications.

# RESULTS AND DISCUSSION

**Conceptual Understanding of Derivatives**

The derivative can be mathematically defined as the slope of the tangent line to a curve at a specific point, allowing us to describe instantaneous rates of change (Orton, 1983). However, students often initially interpret the derivative in a procedural manner without connecting it to practical meaning (Drijvers *et al*., 2016). Researchers have emphasized the importance of contextually-based and conceptually-driven instruction for the derivative, which helps students understand the significance of the mathematical process (Sofronas *et al*., 2011; White, 1961). Approaching the derivative conceptually through visual, physical, and applied perspectives has proven to support student understanding. By providing meaningful applications, we can anchor the abstract concept of the derivative to everyday scenarios and motivate students by demonstrating its relevance (Kember *et al*., 2008). For instance, applications involving velocity, acceleration, or marginal cost/benefit analysis establish tangible connections (Orton, 1983). Everyday contexts related to motion, such as an object's speed or the slope of a hilly road, have been particularly effective in building conceptual understanding of the derivative (White, 1961).

This overview aims to synthesize research-based strategies for teaching the derivative conceptually through practical applications. These strategies focus on visual, physical, and applied problem-solving approaches. Integrating such methods into secondary mathematics instruction has the potential to enhance students' comprehension and their recognition of the real-world significance of this challenging concept in calculus. The overview has briefly discussed some major ways of conceptualizing the derivative.

***Conceptualizing Derivatives through Intuitive Understanding of Slope and Rate of Change***

The conceptualization of derivative through an intuitive understanding of slope and rate of change is a fundamental approach in calculus. Several studies and educational resources support this pedagogical strategy. Dilling & Witzke (2020) emphasizes the importance of connecting derivative to the intuitive concept of slope. They argue that understanding derivative as slopes of tangent lines allows students to develop a more meaningful and coherent understanding of the concept. This approach aligns with the geometric interpretation of derivative, where the slope represents the rate of change at a specific point on the function's graph. Similarly, Thompson (1994) explores the notion of rate of change as a foundational concept for understanding derivative. Thompson suggests that the intuitive understanding of rate of change, such as velocity or growth rates, can serve as a bridge to grasping the concept of derivative. By relating derivative to these familiar real-world contexts, students can develop a deeper conceptual understanding.

Stewart (2015) also emphasizes the intuitive understanding of derivative through slope and rate of change. Stewart presents the graphical interpretation of derivative, highlighting the connection between the slope of a tangent line and the instantaneous rate of change at a point. This graphical approach provides a visual representation that aids students in conceptualizing derivative. Furthermore, Anton *et al*., (2012) emphasis on intuitive understanding of derivative through rate of change. They highlight the interpretation of derivative as rates of change in various contexts, such as motion, population growth, and economics. This approach helps students connect derivative to real-world applications and enhances their conceptual understanding.

Intuitive understanding of slope and rate of change is a widely supported approach in the conceptualization of derivative. Studies by Dilling & Witzke (2020) and Thompson (1994) highlight the importance of connecting derivative to slope and rate of change, while Stewart (2015) and Anton *et al*., (2012) provide instructional resources that emphasize this intuitive approach. For example, Figure 1 describes conceptualizing derivative through a graphical representation using a population growth scenario (i), tangent to the curve at point A (ii) and different positions of tangent lines (iii). The instantaneous rate of change in population at time 3 days when population is 500 can be represented as the slope of the tangent line to the population curve at point (3, 500). Drawing the tangent line relates its slope to the definition of derivative, aiding intuitive understanding. Using population growth anchors the concept. Approximating the slope also reinforces the definition of derivative as a limit. This graphical approach supports conceptualization through visualization and application to a concrete scenario. Thus, establishing connections between slope and rate of change in real-world contexts enables students to cultivate a profound and significant comprehension of this fundamental concept in calculus.

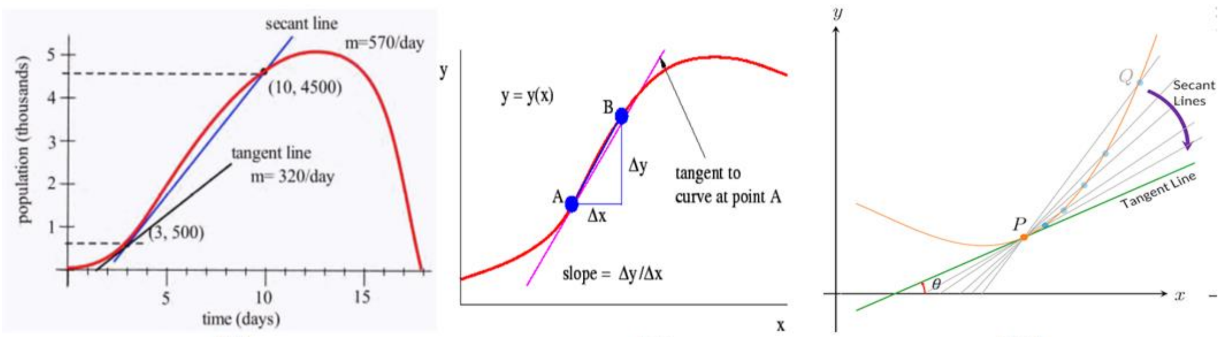


**Figure 1: Graphical Representation of the Curve with Slope and Rate of Change**

***Conceptualizing Derivative through Definitions and Notations***

Definitions of derivatives lay the foundation for students to conceptually understand this important concept of calculus. Derivative can be defined formally as the limit of the average rate of change (Larson & Edwards, 2014). Specifically, the derivatives at a point $x = a$ for a function $f(x)$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Where $h$ approaches zero and the fraction $\frac{f(a+h)-f(a)}{h}$ represents the average rate of change between the points $[a + f(a)]$, and $[a + h, f(a + h)]$ (Thompson, 1994). Connecting this definition to graphical representations of slopes allows students to interpret derivatives visually as the slopes of tangent lines (Orton, 1983).

The derivative of a function $f(x)$ with respect to $x$, written as $f'(x)$, is defined as the limit used to calculate the slope of the tangent line at each point. The domain of the derivative function $f'(x)$ consists of all the $x$-values where this limiting process is defined. Geometrically, $f'(x)$ represents the slope of the tangent line to the graph of $f(x)$ at the point $[x, f(x)]$ (Figure 2).

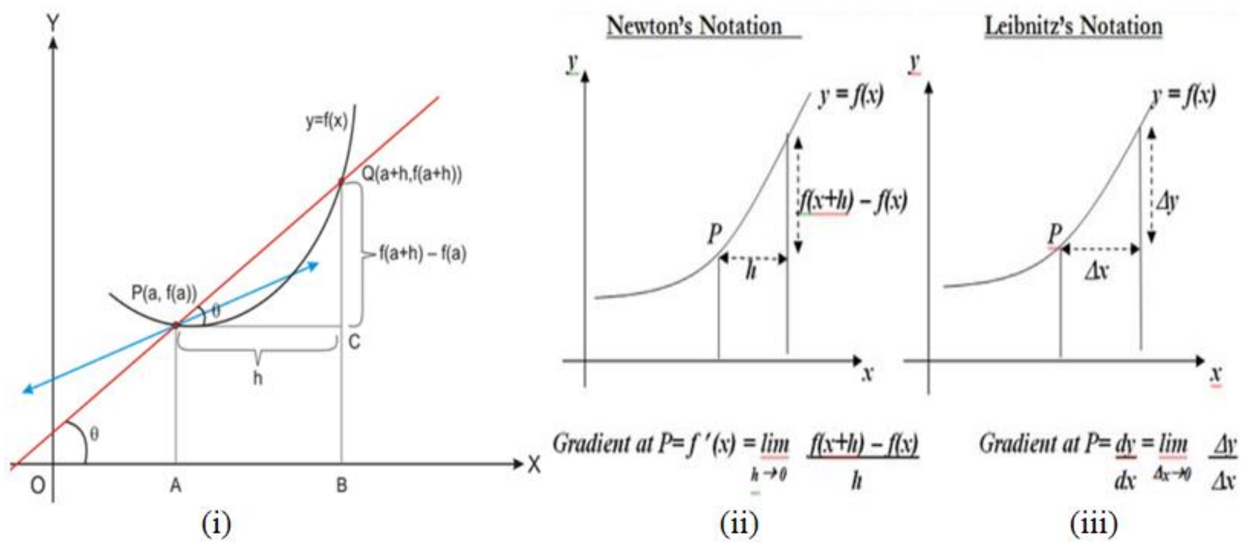


**Figure 2: Graphical Representations of Derivative with Definition and Notations**

In Figure 2, the slope of chord $PQ = \tan \angle QPC = tan\theta = \frac{QC}{PC} = \frac{f(a+h)-F(a)}{h}$. As $Q \rightarrow P$, then $h \rightarrow 0$ and chord $PQ$ will become tangent at $P$.

Notation plays an important role in solidifying conceptual understanding of derivatives. Using derivative notation such as $f'(a)$ emphasizes that the derivative is a function in itself rather than merely the slope at a point (Larson & Edwards, 2014). Symbolic conventions like $\frac{dy}{dx}$ or $\frac{df}{dx}$ (Leibniz notation), $f'(x)$ or $y'$ (Lagrange notation), $\dot{y}$ (Newton notation), $Dy$ (Euler notation), etc. denoting the derivative of a function $y$ with respect to $x$ help students understand derivative as ratios describing instantaneous rates of change (Huang, 2011) (Figure 3).

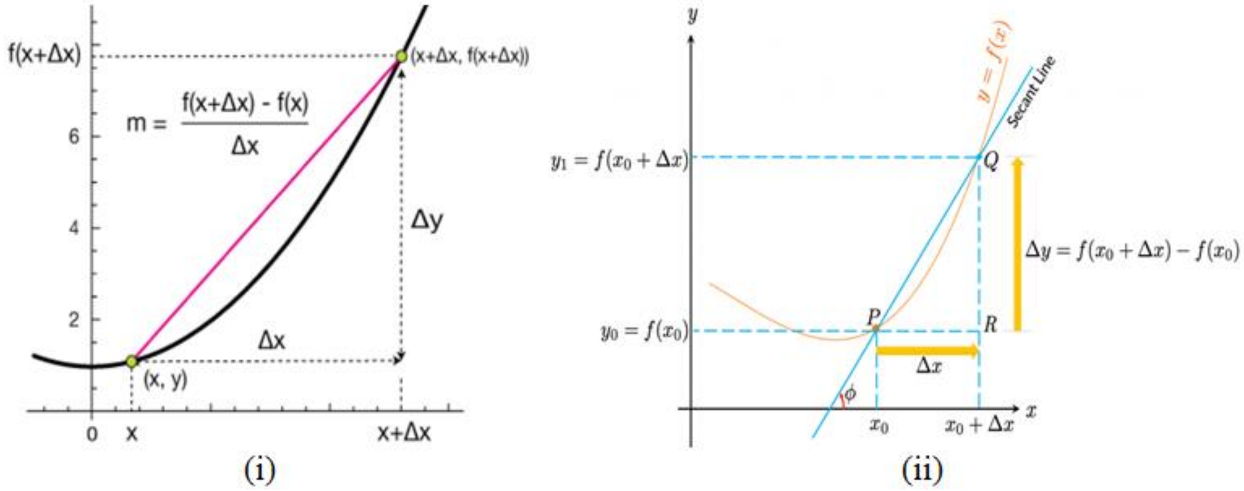


**Figure 3: Graphical Representation of Derivative of a Function with Increment Δx and Δy**

In Figure 3, the slope m of a secant line is written in terms of $f(x)$ at points $x$ and $x + \Delta x$, representing the change in $y$-values over the change in $y$-values $\frac{\Delta y}{\Delta x}$. Imagine that $\Delta x$ approaches zero, where the slope is the exact tangent line rather than a secant approximation. This thought exercise reinforces the definition of derivative as a limiting rate of change, encouraging students to mentally link graphical and analytical slopes involving limits. Relating the derivative to the familiar rise-run ratio also intuitively anchors the concept. This visualization effectively guides students to build an understanding of differentiation from the first principles of limits and graphical interpretations. Exploring how different notations link to graphical, numeric, and real-world interpretations of derivatives aids conceptualization. Overall, formally defining derivatives while highlighting connections to multiple representations supports a coherent conceptual framework (Erens & Eichler, 2015).

***Conceptualizing Derivative through Differentiation Rules***

Memorizing differentiation rules alone does not develop a conceptual understanding of derivatives. However, when accompanied by strong foundations in definition, notation, limits, and graphical/numerical representations, rules can reinforce conceptualizations (Ferrini-Mundy & Graham, 1994). For example, exploring the Power Rule - that the derivative of $x^n$ is $nx^{n-1}$ connects to the definition of derivative as limits of difference quotients and visual representations of slopes (Huang, 2011). Considering how differentiation rules arise from analytic definitions builds coherence (Larson & Edwards, 2014).

Linking rules to multiple perspectives strengthens flexible knowledge. Investigating how graphical behaviors follow from rules, like maximum/minimum points relating to $f'(x) = 0$, supports function thinking (Erens & Eichler, 2015). Applying rules to real-world word problems allows conceptual use beyond symbolic manipulations. Working through detailed derivations of common rules from first principles, like the Chain Rule, helps demystify previously "opaque" procedures (Thompson, 1994). Relating new rules to conceptual schemas forged from definitions/limits enhances understanding of rote drilling (Orton, 1983). Similarly, we can show the relation of the Product Rule and the Quotient Rule. Overall, differentiation rules foster mastery when coupled with rich, linked conceptual foundations (Rassmussen, 2001). Thus, incorporating visual representations during the teaching and learning process, students can develop a deeper conceptual understanding of derivative and differentiation rules. Visualizations help bridge the gap between abstract mathematical concepts and their real-world applications, making the learning experience more engaging and meaningful.

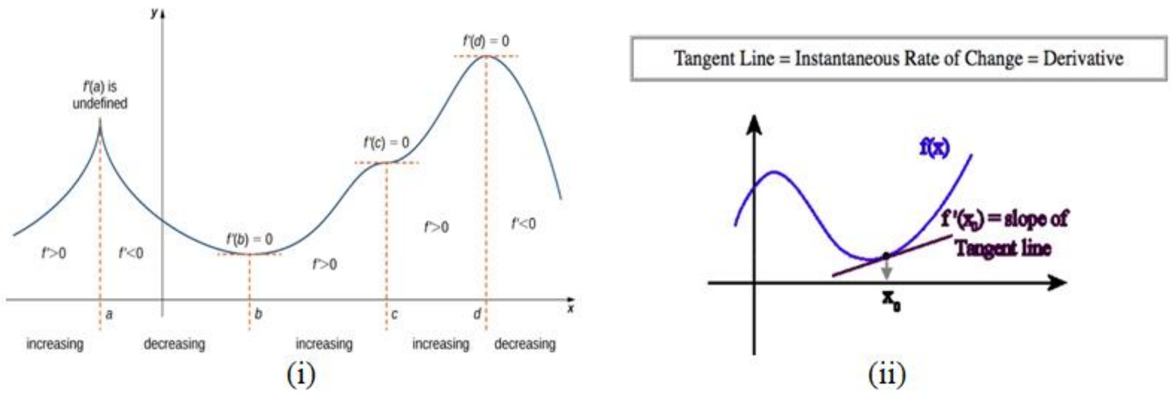


**Figure 4: Graphical Representation of Maxima and Minima and Rate of Change**

In Figure 5, the model presents an example of a chain rule. The composition of functions $f$ and $g$ occurs when the output of $g$ is used as the input for $f$. The range of $g$ must be contained in the domain of $f$ for this composition to work. If $x$ represents the input to $g$, $g(x)$ is the output, which becomes the input to $f$, resulting in $f(g(x))$. The function $h$ represents the transformation from $g'$s input to $f'$s output. The derivative of $h$ measures the change in $h'$s output with a small change in its input, or equivalently, the change in $f$]s output with a change in $g'$s input. The chain rule calculates this derivative by tracing the chain of events from the input of $g$ to the output of $f$. Changing the input to $g$ affects the output of $g$, which in turn affects the input to $f$, resulting in a change in $f'$s output. The derivative of $h'$ is the ratio of the change in the faceted sphere to the change in the sphere. The derivative of $g$ is the ratio of the change in the cube to the change in the sphere, and the derivative of $f$ is the ratio of the change in the faceted sphere to the change in the cube. By multiplying the ratios of the derivatives of $g$ and $f$, the factors corresponding to the change in the cube cancel out, yielding the ratio corresponding to the derivative of h. Using "d" to denote "change in," we can express the chain rule result in terms of the function machine inputs and outputs.

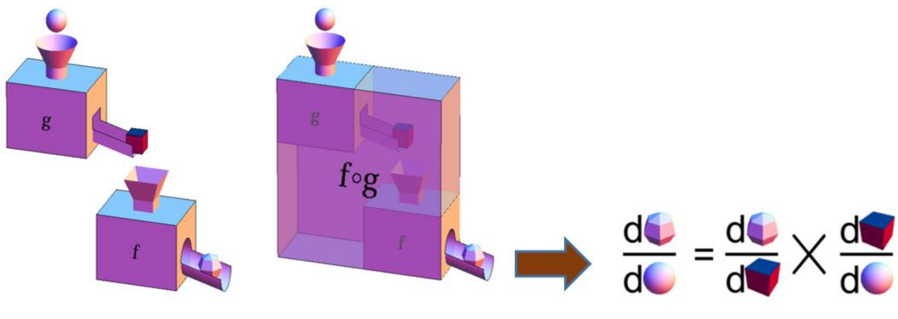


**Figure 5: Graphical Representation of a Model of Chain Rule**

***Conceptualizing Derivative through Concrete Examples and Graphical Representations***

Using concrete examples that emphasize physical phenomena and contextual applications helps students grasp the meaning and significance of derivatives (Jones, 2017). For instance, exploring instantaneous velocity from kinematic equations or acceleration/slopes of position-time graphs fosters intuitive links between concepts (Thompson, 1994). Graphical approaches are essential for visualizing derivatives. Relating the tangent line approximation to the slope of a curve at a point advances understanding of derivatives as rates of change (Orton, 1983). Comparing graphs of functions and their derivative illustrates general properties like maximums relating to where the slope is zero (Erens & Eichler, 2015). Figure 6 presents the graph of concrete examples (i), (ii) and a model of instantaneous speed (iii).

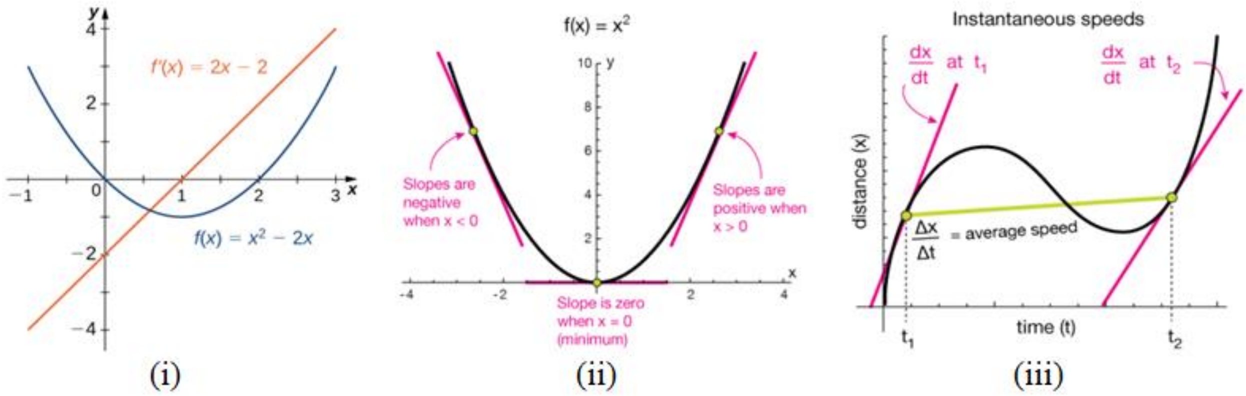


**Figure 6: Graphical Representation of Maxima and Minima and Rate of Change**

The visualization of graphical representation can make more interactive and dynamic exploration of functions and connections between representations using graphing technology (Drijvers *et al*., 2016). Tools like Geometer's Sketchpad, Geogebra, or Desmos let users visualize tangent lines while zooming or manipulating sliders. This dynamic approach promotes flexibility and applied thinking over memorization (Park & Leatham, 2017). Seeing connections between graphs, tables, equations, and descriptions provides reinforcement from complementary perspectives (Huang, 2011). Representational flexibility aids transfer to new contexts or strategy selection for diverse problems (Deliyianni *et al*., 2016). Overall, concrete examples combined with graphical and dynamic visualization fosters intuitive, applicable conceptualization of derivatives. Graphical representations have been shown to support the conceptualization of derivatives (Sofronas *et al*., 2011). Students can analyze graphs to qualitatively reason about rates of change and link graphical behavior to features of the original function (Chauvot, 2009). The identification of increasing and decreasing intervals in a function can be accomplished by analyzing the sign of its derivative over a given interval. The advent of graphing technology has revolutionized the way students can interactively visualize derivatives by manipulating functions and observing instantaneous alterations in both the function's graph and its derivative (Drijvers *et al.*, 2016). This technological advancement provides a dynamic and intuitive platform for students to explore the behavior of functions and deepen their understanding of calculus concepts. As concluded by (Chauvot (2009), the students engaging in graphical analysis tasks promoted conceptual understanding beyond procedural fluency.

Similarly, concrete manipulatives offer another mode for visualizing and building intuition about derivatives (Chatain *et al.*, 2022). For instance, using motion detectors or slinkies to capture velocity or acceleration as a "physical derivative." Such manipulative models have prompted students to reason about quantitative and qualitative attributes of derivatives in applied contexts (Kordosmeier *et al*., 2019). Likewise, real examples of costs and benefits make connections between abstract ideas and real-world use. It provides multiple ways like physical demonstrations to represent the relationships between changes in costs and benefits.

***Conceptualizing Derivative through Employing Real-world Context***

Using real-world contexts helps students understand how derivative models apply to situations (Jones, 2017). Exploring velocity, acceleration, or exponential growth in biology, physics or economics fosters relevance and motivation (Thompson, 1994). Relating derivatives to many practical examples shows their broad utility in modeling rates of tangible processes (Ferrini-Mundy & Graham, 1994). Investigating optimization or related rates problems situated in authentic scenarios promotes active, inquiry-based learning over mere mechanics (Huang, 2011). Graphing calculators or computer programs allow simulations and visualizations that reinforce theoretical understandings with dynamic, interactive representations (Erens & Eichler, 2015). Geometer's Sketchpad or Desmos enables observing changing graphs interactively to build flexible thinking (Drijvers *et al*., 2016). Overall, embedding derivatives within meaningful practical applications provides a purpose for learning beyond standardized exams. This fosters ownership and retention by demonstrating relevance to students' worlds (Jones, 2017).

Solving contextual word problems is critical for students to attach conceptual meaning to derivatives (Sofronas *et al*., 2011). Exercises involving concepts like velocity, acceleration, or profit maximization have engendered an understanding of derivatives as instantaneous rates and tools for optimization. For example, formulating and solving word problems involving real-world scenarios like traffic flow, construction mechanisms of suspension bridges, staircase, etc. can be used to strengthen the conceptualization of derivatives for students. Derivatives, which are mathematical tools that measure rates of change, are actually used in everyday applications instead of just abstract calculations (Yang, 2016). Figure 7 presents the different real-world models for conceptualizing derivatives.

**Figure 7: Graphical Representation of Images in the Real-world Context**

**Defining Derivative**

The derivative of a function $f(x)$ at a point $x$, denoted by $f'(x)$, is defined as the limit of the average rate of change of $f$ as $\Delta x$ approaches 0. It represents the instantaneous rate of change of the function with respect to the independent variable (Stewart, 2015). Derivatives allow mathematicians to analyze rates of change, optimal values, linearization, and other properties that help understand naturally occurring phenomena (Erens & Eichler, 2015). They are foundational concepts in calculus and underlie principles of physics, engineering, economics, and other quantitative disciplines (National Research Council, 2012). Historically, the development of derivative concepts was fundamental to the emergence of calculus and its applications across sciences. This established much of the foundation for modern quantitative fields. (Boyer & Merzbach, 1989). Derivative can be defined from various perspectives.

Geometrically, the derivative indicates the slope of the tangent line to the function's graph at any point (Figure 1). Symbolically, derivative gives algebraic rules to determine instantaneous velocities and incremental change from functions defined by formulas (Huang, 2011) (Figure 2(ii). Graphically, derivatives permit identifying maxima, minima, and points of inflection to optimize target quantities (Figure 4(i). Their uses range from modeling simple kinematic situations to solving complex optimization problems (Hyer & Gardner, 2007). Computationally, derivatives establish procedures for taking derivatives of basic and composite functions using rules like the power, constant multiple, sum, difference, and quotient rules. This allows analysts to mathematically model real-world situations (Larson & Hostetler, 1994). Conceptually, derivative formalizes the intuitive notion of instantaneous rate of change and operationalizes it mathematically. This links calculus to physical phenomena and geometric relationships (Orton, 1983). Pedagogically, derivatives are a threshold concept that students must master to progress in calculus. Research finds transforming student preconceptions is crucial for comprehensive understanding (Lobato *et al*., 2012).

**Strategies for Conceptualizing Derivative**

Graphical approaches play a crucial role in interpreting derivatives as rates of change from geometric and numeric perspectives, aiding in the development of intuitions for derivatives as slopes of tangent lines. Dynamic graphing technology facilitates interactive exploration of how derivatives change with functions (Drijvers *et al*., 2016). Real-world connections enhance conceptual understanding by applying derivatives to contextual problems involving velocity, acceleration, maximization, and other applications promoting student engagement (Schwalbach & Dosemagen, 2010).

Utilizing multiple representations simultaneously, including numeric, graphical, symbolic, and verbal forms, supports translational reasoning between different representations (Huang, 2011). Multidimensional perspectives reinforce accurate conceptual models (Brown, 2019). Limit-based definitions formalize derivative formulas, while numeric and graphical approaches provide intuitive understanding (Orton, 1983). Bridging intuitive and formal aspects fosters more robust cognition of this essential concept (Drijvers *et al.*, 2016). Tangible manipulatives, such as simulations of motion, rates of slope, stretching/shrinking, and embodied metaphors, make derivatives more tangible and enhance conceptual understanding (NCTM, 2000). Physical experiences further solidify conceptual understanding (Chauvot, 2009). Active learning approaches prioritize conjecture, justification, and collaborative sense-making over routine skill-building (Lobato *et al*., 2012). Inquiry-based activities promote deeper engagement compared to traditional instruction (Khasawneh *et al*., 2023).

Several strategies can be employed to conceptualize derivatives effectively. Problem-solving contexts, involving real-world scenarios, require the application of derivative concepts and rules, improving understanding of their practical uses. Verbal explanations, where students articulate and reflect on their derivative ideas, aid in identifying and addressing misconceptions (Chauvot, 2009). Metaphors and analogies, relating derivatives to familiar actions like velocity or units flowing in/out, facilitate initial comprehension of rates of change and sustain interest (Lakoff & Nuñez, 2000). Modeling mathematical thinking by thinking aloud during derivative problem-solving demonstrates expert reasoning processes, allowing students to internalize holistic approaches (Lobato *et al*., 2012). Identifying and addressing students' pre-existing ideas and misconceptions helps frame instruction relative to their zones of proximal development (Confrey, 1990). Technology, such as dynamic function graphs, sliders, and interactive simulations, provides dynamic and visual tools for reasoning about derivatives without relying heavily on symbolic manipulation (Yang & Baldwin, 2020). Some major strategies for conceptualizing derivatives have been discussed below.

***Intuitive Understanding of Slope and Rate of Change***

The slope is intuitively grasped from observing hills, where steepness matches rise over run. This informal sense of faster or slower change provides a basis to define the rate. Rate of change can be used in everyday activities like driving distances over time, and linking informal ideas to symbols. Comparing scale diagrams nurtures seeing constant rates as straight lines (Lakoff & Nuñez, 2000). Research has shown that when it comes to understanding slope, gestures can convey meaning before formal mathematical symbols, indicating that physical understanding comes before the development of formal skills (Rasmussen *et al*., 2004). Mathematics builds upon qualitative observations of changing quantities in various contexts, as these experiences serve as a foundation for further mathematical understanding. Intuition develops in other ways as well. For example, comparing the crawling speeds of babies and adults can provide an intuitive feel for slopes (Lakoff & Nunez, 2000). Real-world examples such as the steepness of hiking trails or the slowing down of rolling balls connect rates to tangible objects (Orton, 1983). Judging winners in a footrace involves comparing changes, and this kind of informal problem-solving precedes the use of mathematical symbols (Confrey, 1990). Experiences like these help students develop an intuitive understanding of concepts like slope and derivative, which are later expressed more precisely in mathematical terms.

***Visualizing Derivative through Graphs and Geometric Interpretations***

The slope of a tangent line visually shows the instantaneous rate of change or derivative at a point (Orton, 1983). As the interval near a point narrows, the chord slope approaches the tangent slope. Graphing polar functions allows seeing velocity and direction changes. Relating derivatives to tangent slopes helps understand kinematics concepts like acceleration (Erens & Eichler, 2015). The area under curves intuitively links to accumulated quantities over time, supporting interpreting derivative as rates given by slopes (Lobato *et al*., 2012). Gaining an understanding of derivatives as limits and their practical optimization applications becomes clearer when observing tangent lines representing small changes (Thompson & Silverman, 2008). Manipulatives like slinky springs show derivative geometrically through distance changes over time (Larson & Farber, 2004). Motion maps overlaying graphs develop system thinking for modeling dynamics (Sherin, 2001). Graphs help visualize derivatives through techniques like manipulating sliders online (Drijvers *et al*., 2016), connecting areas to changes, and animating tangent lines (Heid, 1988).

***Exploring the Concept of Instantaneous Rates of Change of a Curve***

The idea of an instantaneous rate of change arises from considering what happens as the measurement interval approaches zero (Thompson, 1994). Exploring varying intervals on motion graphs intuitively develops the derivative as the limiting average rate (Ferrini-Mundy & Graham, 1994). Graphing calculator software displaying chords and tangents fosters examining chord approximations converging to the limit. Manipulatives like marbles on ramps relate physical rates to their graphical representation as slopes (McDermott *et al*., 1987). Comparing tables, graphs, and rules coordinates multiple representations for more coherent conceptualization (Orton, 1983). Additional conditions can also help explore instantaneous rates. Actively manipulating graphs with sliders dynamically observes how slopes change and form the tangent line (Drijvers *et al*., 2016). Relating qualitative observations of changing quantities over infinitesimal intervals to limited definitions links intuitions and formalism (Davis & Vinner, 1986). Qualitatively comparing speeds in contexts like racing supports instantaneous ideas (Confrey & Smith, 1995). Relating zooming microscopically to the limit process enriches physical intuitions (Thompson et al., 2014). These various representations foster exploring the instantaneous rate concept.

***Applying the Concept of Limits to Define Derivative***

The formal definition of the derivative is based on the limiting process of the average rate of change over narrowing intervals approaching zero, as described by Orton (1983). This concept can be explored by examining rates over varying intervals to develop intuition. Using graphing calculators to compute chord slopes and visualize their convergence to the tangent slope reinforces the limited interpretation of the derivative. It is critical to relate quantitative and qualitative notions of instantaneous rate to the limit definition in order to foster coherent understanding, as emphasized by Ferrini-Mundy and Graham (1994). This understanding becomes particularly important when analyzing examples where average and instantaneous rates differ, such as at corners, since limits are necessary to precisely quantify such behavior (Drijvers *et al*., 2016). Investigating limits numerically and graphically before addressing them symbolically helps students connect intuitions to formal definitions (Huang, 2011). Using different representations can help with the concepts. Looking at limits, rates of change, and slopes together connects them (Thompson, 1994). Other ideas support applying limits to derivatives. Students can watch chords move closer to tangents using software (Erens & Eichler, 2015). Physical examples where things get closer as changes shrink give meaning to limits (Larson & Farber, 2004). Engaging in numerical calculations to approximate limits adds a concrete dimension to the concept (Swinyard, 2011). Comparing derivatives of commonly encountered functions helps solidify understanding (Jones, 2017). Employing multiple perspectives to comprehend these ideas reinforces the fundamental definition of the derivative as a limit.

***Interpretation of Derivative***

Derivatives have various interpretations that shed light on their meaning and significance. The derivative can be interpreted and understood from various perspectives that build students' conceptualization. From a rate of change view, it represents the instantaneous slope of the tangent line at a point on a function's graph (Orton, 1983). Geometrically, this connects to visualizing the derivative as the slope of the tangent line. Derivatives also model real-world contexts through physical interpretations like velocity or profit rates (Thompson, 1994). Numerically, it describes how quickly a function is increasing or decreasing (Ferrini-Mundy & Graham, 1994). Analytic uses involve finding stationary points and inflection points. Procedurally, students apply derivative rules and formulas computationally or through implicit differentiation of more complex functions (Drijvers *et al*., 2016; Jones, 2017). Locally, it approximates linear behavior near a point and depicts incremental changes for optimization and sketching (Larson & Edwards, 2014). Approximations in mathematics often utilize tangents represented in linear form (Drijvers *et al*., 2016). Symbolic calculations involve interpreting rules and following specific steps (Huang, 2011). However, to achieve flexible application, it is crucial to develop a comprehensive conceptual understanding that connects different representations (Thompson, 1994).

**Applications of Derivatives in Daily Life for Secondary-Level Mathematics Students**

Secondary students encounter numerous real-life applications of derivatives in their day-to-day experiences, which can serve as a source of motivation for learning about derivatives (Thompson & Silverman, 2008). Here are some brief examples of real-life applications that secondary-level students may encounter:

***Physics***

Derivatives are used to analyze the motion of objects, such as calculating velocity, and acceleration, and determining the behavior of moving objects (cars, projectiles, etc.). They are fundamental in understanding concepts like kinematics and dynamics (Larson & Edwards, 2014).

***Sports and Athletics***

Derivative is used in sports analysis to study the performance of athletes. It can be applied to analyze factors like speed, acceleration, and optimal strategies in sports such as track and field, swimming, and cycling (Hyer & Gardner, 2007).

***Economics and Business***

Derivatives play a crucial role in economics and finance. It is used to model and analyze cost, revenue, and profit functions, optimize production and pricing strategies, and understand the behavior of financial markets (Orton, 1983; Rasmussen, 2001). It is also used to analyze investment strategies and calculate interest rates.

***Engineering***

Derivatives can be used in various engineering fields to analyze rates of change, optimize designs, speed, and efficiency of machines, and understand the behavior of physical systems. They are applied in areas such as mechanical engineering, electrical engineering, and civil engineering.

***Medicine and Biology***

Derivative is used in medical and biological research to model physiological processes, model the spread of diseases, analyze drug concentration in the body, and understand physiological processes (Drijvers *et al*., 2016). It is used to analyze the rate of change in biological systems and study population dynamics. They are used in fields such as pharmacology, epidemiology, and biomechanics.

***Computer Science***

In the realm of computer science, the derivative finds valuable applications, especially in domains like machine learning, data analysis, and optimization algorithms. It plays a key role in optimizing algorithms, analyzing data structures, and comprehending the behavior of computer programs (Huang, 2011). By leveraging the power of derivatives, computer scientists can enhance the efficiency and effectiveness of computational systems, enabling advancements in various technological fields.

***Environmental Science***

The derivative serves as a valuable tool for modeling and analyzing environmental systems, encompassing areas such as the spread of pollutants, examining rates of change in ecological dynamics, and assessing the impacts of climate change. By employing derivatives, researchers and scientists gain insights into the intricate workings of environmental processes, enabling a better understanding and prediction of complex phenomena (Thompson, 1994). This knowledge is crucial for developing effective strategies for environmental management, conservation, and mitigating the effects of environmental changes.

These are just a few examples of the wide-ranging applications of derivatives in daily life. Derivatives can be used in various fields to analyze rates of change, optimize processes, and understand the behavior of systems. It is used as a valuable tool for modeling and solving real-world problems. These applications can help students see the relevance and practicality of derivatives in their daily lives, making the concepts more meaningful and engaging. Teachers can incorporate other more real-life examples and problem-solving tasks to illustrate how derivative is used in various fields.

# CONCLUSION

Building a strong foundation for derivatives is important. Prioritizing meaningful learning of foundational derivative concepts sets the stage for success in STEM and quantitative fields relying heavily on mathematical modeling and problem-solving skills. It is needed to develop a strong conceptual foundation of derivatives early in mathematics courses. Visualizing rates of change geometrically and intuitively grasping instantaneous behavior through limits lays the groundwork for future success. As mathematics progresses to more advanced topics relying on calculus, such as physics, engineering, and economics, a coherent and flexible understanding of derivatives cannot be underestimated. It is necessary to explore multiple linked representations of derivatives and build relationships between graphical, numeric, contextual, and symbolic aspects. Investigations involving real-world applications, graphical technologies, and physical models can enrich students' intuitions before introducing formal definitions. Ongoing assessments should evaluate conceptual understanding over rote manipulation of procedures to guide necessary remediation.

There are several strategies that can be employed to build students' conceptual understanding of derivatives. Visualizing rates of change geometrically through graphs of functions and relating the slope of tangents to real-world contexts like speed and velocity lays an intuitive foundation. The formal definition of the derivative can be connected to intuitions about average and instantaneous change rates through limiting processes while approximating the derivative numerically and graphically. Developing understanding across multiple linked representations from graphical and numeric to analytic, verbal, and contextual perspectives reinforces comprehension as students translate between representations. Relating derivatives to experiences with magnification and interpreting them as describing the best linear approximations near a point also fosters intuition. Working through physical examples involving motion, costs, growth, and other relatable phenomena motivates learning by connecting mathematical concepts to real-world situations. Exploring functions through finding extrema, and inflection points, and investigating concavity while generalizing across diverse examples deepens conceptual insights. Procedural fluency with derivative rules is strengthened when algorithms and symbolic manipulations are linked tightly to their conceptual underpinnings. With a focus on building rich, flexible conceptual schemas through visual, numeric, contextual, and multi-representational perspectives, students can develop a strong foundation for success in calculus.

It is recommended that curricula should allow for rich investigations of derivative concepts through visual, numeric, and applied contexts before introducing symbolic rules and algorithms. Students need time to develop intuitions about rates of change and their real-world interpretations through qualitative observations and technological explorations. Formal definitions are best introduced after intuitions have been built, making abstract concepts more meaningful and accessible. A greater emphasis on conceptual comprehension over procedural fluency in early learning can provide students with a stronger conceptual foundation to successfully apply and generalize derivative concepts.

**Conflicting Interests**

The authors of this article declare no potential conflicts of interest about the research, authorship, and publication of the article.

# REFERENCES

- Anton, H., Bivens, I., & Davis, S. (2012). *Calculus: Early transcendentals* (10th ed.). John Wiley & Sons.
- Atit, K., Uttal, D. H. & Stieff, M. (2020). Situating space: Using a discipline-focused lens to examine spatial thinking skills. *Cognitive Research, 5*(1), 1-16. https://doi.org/10.1186/s41235-020-00210-z
- Bell, E. T. (2008). *The development of mathematics*. Courier Corporation.
- Boyer, C. B. (1968). *A history of mathematics*. Wiley.
- Boyer, C. B., & Merzbach, U. C. (1989). *A history of mathematics*. Wiley.

- Brown, J. P. (2019). Real-world task context: Meanings and roles. In: Stillman, G., Brown, J. (eds), *Lines of inquiry in mathematical modelling research in education.* ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-030-14931-4_4
- Chatain, J., Ramp, V., Gashaj, V., Fayolle, V., Kapur, M., Sumner, R., & Magnenat, S. (2022). *Grasping derivatives: Teaching mathematics through embodied interactions using tablets and virtual reality*. Proceedings of the 21st Annual ACM Interaction Design and Children Conference June 2022. Pages 98–108. https://doi.org/10.1145/3501712.3529748
- Chauvot, J. B. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator-researcher. *Teaching and Teacher Education: An International Journal of Research and Studies*, *25*(2), 357-370.
- Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. *Review of Research in Education, 16,* 3-56. https://doi.org/10.3102/0091732X016001003
- Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjectured learning trajectories. In Kelly, A. & Lesh, R. (Eds.), *Handbook of research design in mathematics and science education* (p. 231–265). Lawrence Erlbaum Associates Publishers.
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66-86. https://doi.org/10.2307/749228
- Cornu B (1991) Limits. In: Tall D (ed), *Advanced mathematical thinking*. Dordrecht, pp 153–166, Kluwer.
- Cullen, C. (1996). Astronomy and mathematics in ancient China: The Zhou bi Suan Jing. *Journal for the history of astronomy, 29*(96), 553-578.
- Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *The Journal of Mathematical Behavior*, *5*(3), 281–303.
- Deliyianni, E., Gagatsis, A., Elia, I. (2016). Representational flexibility and problem-solving ability in fraction and decimal number addition: A structural model. *International Journal of Science and Mathematics Education 14* (Suppl 2), 397– 417. https://doi.org/10.1007/s10763-015-9625-6
- Dilling, F., Witzke, I. (2020). The use of 3D-printing technology in calculus education: Concept formation processes of the concept of derivative with printed graphs of functions. *Digital Experiences in Mathematics Education*, *6*, 320–339. https://doi.org/10.1007/s40751-020-00062-8
- Drijvers, P., Ball, L., Barzel, B., Heid, M.K., Cao, Y., Maschietto, M. (2016). Uses of technology in lower secondary mathematics education. In: *Uses of technology in lower secondary mathematics education. ICME-13 topical surveys*. Springer, Cham. https://doi.org/10.1007/978-3-319-33666-4_1
- Edwards, C. H. (1979). *The historical development of the calculus*. Springer. https://doi.org/10.1007/978-1-4612-6230-5
- Erens, R., Eichler, A. (2015). The use of technology in calculus classrooms – Beliefs of high school teachers. In: Bernack-Schüler, C., Erens, R., Leuders, T., Eichler, A. (eds), *Views and beliefs in mathematics education. Freiburger empirische forschung in der mathematikdidaktik*. Springer Spektrum, Wiesbaden. https://doi.org/10.1007/978-3-658-09614-4_11
- Ferrini-Mundy, J., & Graham, K. G. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning* (pp. 31–45). Mathematical Association of America.
- Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25. https://www.learntechlib.org/p/139005/.
- Huang, C.-H. (2011). Engineering students' conceptual understanding of the derivative in calculus. *World Transactions on Engineering and Technology Education, 9*(4), 209-214.
- Hyer, C., & Gardner, A. (2007). *Discovering the derivative can be invigorating: Mark's journey to understanding instantaneous velocity*. Theses and Dissertations, 1173. Brigham Young University. https://scholarsarchive.byu.edu/etd/1173
- Jones, S. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior, 45*, 95-110. https://doi.org/10.1016/j.jmathb.2016.11.002.
- Joseph, G. G. (2000). *The crest of the peacock: Non-European roots of mathematics*. Penguin.
- Katz, V. J. (Ed.). (1998). *A history of mathematics: An introduction*. Addison-Wesley.
- Kember, D., Ho, A., & Hong, C. (2008). The importance of establishing relevance in motivating student learning. *Active Learning in Higher Education, 9,* 249-263. https://doi.org/10.1177/1469787408095849.
- Khasawneh, E., Hodge-Zickerman, A., York, C. S., Smith, T. J., & Mayall, H. (2023). Examining the effect of inquiry-based learning versus traditional lecture-based learning on students' achievement in college algebra. *International Electronic Journal of Mathematics Education, 18*(1), em0724. https://doi.org/10.29333/iejme/12715
- Lakoff, G., & Nunez, R. (2000). *Where mathematics comes from*. Basic Books.
- Larson, R., & Edwards, B. H. (2014). *Calculus* (10th ed.). Cengage Learning.
- Larson, R., & Farber, B. (2004). *Elementary linear algebra*. Pearson Education.
- Larson, R., & Hostetler, R. P. (1994). *Calculus*. Houghton Mifflin.
- Larson, R., Edwards, B., & Hostetler, R. (2013). *Calculus of a single variable* (10th Ed.). Cengage Learning.

- Lobato, J., & Rhodehamel, B., & Hohensee, C. (2012). Noticing as an alternative transfer of learning process. *The Journal of the Learning Sciences, 21*(3), 433-482. https://doi.org/10.1080/10508406.2012.682189.
- Maltese, A. V., & Tai, R. H. (2010). Eyeballs in the fridge: Sources of early interest in science. *International Journal of Science Education, 32*(5), 669-685, https://doi.org/10.1080/09500690902792385
- McDermott, L. C., Rosenquist, M. L., & van Zee, E. H. (1987). Student difficulties in connecting graphs and physics: Examples from kinematics. *American Journal of Physics, 55*(6), 503-513. https://doi.org/10.1119/1.15104
- National Research Council. (2012). *Education for life and work: Developing transferable knowledge and skills in the 21st century*. National Academies Press.
- NCTM. (2000). *Principles and standards for school mathematics*. Author.
- NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.
- Netz, R. (2022). The generation of archimedes. In *A new history of Greek mathematics* (pp. 107-220). Cambridge: Cambridge University Press. https://doi.org/10.1017/9781108982801.004
- Orton, A. (1983). Students' understanding of differentiation. *Educational studies in mathematics, 14*, 235–250. https://doi.org/10.1007/BF00410540
- Park, J., & Leatham, K. (2017). The role of contextualized tasks and multiple representations in enhancing students' understanding of the derivative concept. *International Journal of Science and Mathematics Education, 15*(2), 285-306.
- Rasmussen, C., Stephan, M., & Keene, K. (2004). Classroom mathematical practices and gesturing. The *Journal of Mathematical Behavior, 23*, 301-323. https://doi.org/10.1016/j.jmathb.2004.06.003.
- Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *The Journal of Mathematical Behavior. 20*(1), 55–87. https://doi.org/10.1016/S0732-3123(01)00062-1.
- Schwalbach, E., & Dosemagen, D. (2010). Developing student understanding: Contextualizing calculus concepts. *School Science and Mathematics, 100*, 90 - 98. https://doi.org/10.1111/j.1949-8594.2000.tb17241.x.
- Schwartz, L. (1950). Théorie des distributions. *Annales de l'Institut Fourier*, *7*, 1-141.
- Sherin, B. L. (2001). How students understand physics equations. *Cognition and Instruction*, *19*(4), 479–541. https://doi.org/10.1207/S1532690XCI1904_3
- Sofronas, K., & DeFranco, T., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior, 30*, 131-148. https://doi.org/10.1016/j.jmathb.2011.02.001.
- Stewart, J. (2015). *Calculus: Concepts and contexts* (4th Ed.). Cengage Learning.
- Stillwell, J. (2010). *Mathematics and its history*. Springer Science & Business Media.
- Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *The Journal of Mathematical Behavior, 30*, 93-114. https://doi.org/10.1016/j.jmathb.2011.01.001.
- Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.
- Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2-3), 229-274. https://doi.org/10.1007/BF01273664.
- Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Mathematical Association of America.
- Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing, WISDOMe Monographs* (Vol. 4, pp. 1-24). University of Wyoming.
- Wewe, M. (2020). The profile of students' learning difficulties in concepts mastery in calculus course. Decimal. *Journal Matematika, 3*(2), 161-168. https://doi.org/10.24042/djm.v3i2.6421.
- White, A. R. (Ed.). (1961). *The principles of philosophy: Part 2, De mundo: Translation, with introduction and notes by I. Bernard Cohen and Anne Whitman*. Martinus Nijhoff.
- Yang, D. & Baldwin, S.J. (2020). Using technology to support student learning in an integrated STEM learning environment. *International Journal of Technology in Education and Science (IJTES), 4*(1), 1-11.
- Yang, Xin-She. (2016). *Engineering mathematics with examples and applications*. Academic Press.