ISSN 2664-4150 (Print) & ISSN 2664-794X (Online)
South Asian Research Journal of Engineering and Technology
Abbreviated Key Title: South Asian Res J Eng Tech

| Volume-1 | Issue-2| Aug-Sep -2019 | **DOI: 10.36346/SARJET.2019.v01i02.003**

Original Research Article

Product of Polycyclic-by-Finite Groups (PPFG)

Behnam Razzaghmaneshi*
Assitant Professor, Department of Mathematics and Computer science, Islamic Azad University Talesh Branch, Talesh, Iran

***Corresponding Author**
Behnam Razzaghmaneshi

Article History
Received: 20.09.2019
Accepted: 27.09.2019
Published: 30.09.2019

Abstract: In this paper we show that If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Keywords: polycyclic-by-finite, soluble group, maximal condition, finite group AMS classification: 2oF32.

INTRODUCTION

In 1955 N. Itô (see [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (see[21]) and L.Redei (1950)(see [22]) considered products of cyclic groups, and around 1965 O.H.Kegel (See [30, 31]) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition. He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [20, 1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See [20])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1-4] and [6]) , N.S. Chernikov (see [5]), S. Franciosi, F. de Giovanni (see [3, 6, 32-36]), O.H.Kegel (see [8]), J.C.Lennox (see [12]) , D.J.S. Robinson(see [9] and [15]), J.E. Roseblade (see [13]), Y.P.Sysak(see [37-40]), J.S.Wilson (see [41]), and D.I.Zaitsev(see [11] and [18]).

Now, In this paper we show that If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Priliminaries: (elementary properties and theorems.)

Difinition: Recall that the FC-centre of a group G is the subgroup of all elements of G with a finite number of conjugates. A group is an FC-group if it coincides with its FC-centre.

Lemma: Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized. Moreover, every term of the upper central series of G is factorized.

Proof: Since S is factorized, we have that S=$(A\cap S)(B\cap S)$. Let x=ab be an element of S, where a is in $A\cap S$ and b is in $B\cap S$ .If c=a_1b_1 is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1,x]=[a_1,ab]=[a_1,b]=[cb_1^{-1},b]=[c,b]^{b_1^{-1}}=1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of [4]. In particular, the center of G is factorized. It follows from Lemma 1.1.2 of [4] that also every term of the upper central series of G is factorized.

Lemma: Let the group G=AB be the product of two subgroups A and B. If A_1, B_1, and F are the FC-centers of A, B, and C, respectively, then $F=A_1F\cap B_1F$. In particular, if A and B are FC-groups, the FC-centre of G is factorized subgroup.

Proof: Let x be an element of $A_1F\cap B_1F$, and write x=au where a is in A_1 and u is in F. Since the centralizers $C_A(a)$ and $C_A(u)$ have finite index in A, the index $|A: C_A(x)|$ is also finite. Similarly, $C_B(x)$ has finite index in B. Therefore $|G:\langle C_A(x),C_B(x)\rangle|$ is finite by Lemma 1.2.5 of [4]. It follows that $C_G(x)$ has finite index in G and hence x belongs to F. Thus $F=A_1F\cap B_1F$.

Lemma: (See [7]) Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof: Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.11 have $Z(G)=(A\cap Z(G))(B\cap Z(G))=1.$ The centralizer $C=C_G(A\cap C_G(G'))\ contains\ AG',$ and so is normal in G. Since $B\cap(AZ(C))\le Z(G)=1,$ it follows that $AZ(C)=A(B\cap AZ(C))=A.$ This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.9, we have $A\cap G'\le A\cap C_G(G')\le Z(C)=1.$

Similarly $B\cap G'\le B\cap C_G(G')\le Z(C)=1.$ The factorizer $X=X(G')$ has the triple factorization $X=A^*B^*=A^*G'=B^*G'$, Where $A^*=A\cap BG'\ \ and\ \ B^*=B\cap AG'.$ Thus X is nilpotent by Corollary 2.8, so that

$$Z(X)=(A\cap Z(X))(B\cap Z(X))$$

is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G']\le N\cap G'\le A\cap G'=1.$ Therefore we obtain the contradiction $N\le A\cap G_G(G')=1.$

Corrollary: Let the finite group $G=A_1\ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

Proof. Let p be a prime, and for every i=1…,t let P_1 be the unique Sylow p-complement of A_i. If $i\neq j$, the subgroup A_iA_j is soluble by Theorem 2.4.3 of [4]. Hence it follows from Lemma 2.6,that P_iP_j is a Sylow p-complement of A_iA_j. Thuse the subgroups $P_1,\ldots,P_t$ pairwise permute, and the product $P_1P_2\ldots P_t$ is a Sylow p-complement of G. Since G has a Sylow p-complement for every prime p, it is soluble.

Theorem (See [8, 10])**:** If the finite group G=AB is the product of two nilpotent subgroups A and B, then G is soluble.

Proof: See [4], (Theorem 2.4.3).

Lemma: Let A and B be subgroups of a group G, and let A_1 and B_1 be subgroups of A and B, respectively, such that $|A:A_1|\le m$ and $|B:B_1|\le n.$ Then $|A\cap B:A_1\cap B_1|\le mn.$

Proof : To each left coset $x(A_1\cap B_1)$ of $A_1\cap B_1$ in $A\cap B$ assign the pair of left cosets $(xA_1,xB_1).$ Clearly this defines an injective map from the set of left cosets of $A_1\cap B_1$ in $A\cap B$ into the cartesian product of the set of left cosets of A_1 in A and the set of left cosets of B_1 in B. The lemma is proved.

Lemma (See [11])**:** Let the finitely generated group G=AB=AK=BK be the product of two ablian-by-finite subgroups A and B and an abelian normal subgroup K of G. Then G is nilpotent-by-finite.

Proof: Let A_1 and B_1 be abelian subgroups of finite index of A and B, respectively, and let n be a positive integer such that $|A:A_1| \leq n$ AND $|B:B_1| \leq n$. Since G is finitely generated, it has only finitely many subgroups of each finite index, and hence the intersection H of all subgroups of G with index at most n^4 also has finite index in G. In particular H is finitely generated.

Consider a finite homomorphic image H/N of H. Then N has finite index in G, and hence also its core N_G has finite index in G. Let $p_1,\ldots,p_t$ be the prime divisors of the order of the finite abelian group $K/(K \cap N_G)$. For each $j \leq t$, let $K_j/(K \cap N_G)$ be the $p'_j - component$ of $K/(K \cap N_G)$. Clearly each K, is normal in G and $\bigcap_{j=1}^{t} K_j = K \cap N_G$. The factor group $\overline{G} = G/K,$ has the triple factorization $\overline{G} = \overline{A}\overline{B} = \overline{A}\overline{K} = \overline{B}\overline{K}$,where $\overline{K}$ is a finite normal $p_j - subgroup$ of $\overline{G}$. Clearly

$$|\overline{G} : \overline{A} \cap \overline{B}| = |\overline{G} : \overline{A}|.|\overline{A} : \overline{A} \cap \overline{B}| = |\overline{G} : \overline{A}|.|\overline{G} : \overline{B}|$$

$$= |\overline{K} : \overline{A} \cap \overline{K}|.|\overline{K} : \overline{B} \cap \overline{K}| = p_j^k$$

for some non-negative integer k. On the other hand, $|\overline{A} \cap \overline{B} : \overline{A}_1 \cap \overline{B}_1| \leq n^2$ by Lemma 2.16, so that $|\overline{G} : \overline{A}_1 \cap \overline{B}_1| \leq p_j^k n^2$. As $\overline{A}_1$ and $\overline{B}_1$ are abelian, the intersection $\overline{A}_1 \cap \overline{B}_1$ is contained in the centre of $\langle \overline{A}_1, \overline{B}_1 \rangle$, and the factor group $\langle \overline{A}_1, \overline{B}_1 \rangle / (\overline{A}_1 \cap \overline{B}_1)$ has order at most $p_j^k n^2$. Let $\overline{P}/(\overline{A}_1 \cap \overline{B}_1)$ be a Sylow p_j-subgroup of $\langle \overline{A}_1, \overline{B}_1 \rangle / (\overline{A}_1 \cap \overline{B}_1)$. Then $|\langle \overline{A}_1, \overline{B}_1 \rangle : \overline{P}| \leq n^2$, and since $|\overline{G} : \langle \overline{A}_1, \overline{B}_1 \rangle| \leq n^2$ by Lemma 2.2, we obtain $|\overline{G} : \overline{P}| \leq n^4$. Therefore HK_j/K_j is contained in $\overline{P}$ .As an extension of the central subgroup $\overline{A}_1 \cap \overline{B}_1$ by a finite p_j-group, $\overline{P}$ is nilpotent, so that $H/(H \cap K_j) \simeq HK_j / K_j$ is also nilpotent for each j. Hence.

$H \Big/ \left(\bigcap_{j=1}^{t} (H \cap K_j) \right) = H/(K \cap N_G)$ is nilpotent. We have shown that each finite homomorphic image of H is nilpotent . As K is abelian, H is soluble, and hence even nilpotent (Robinson 1972, Part 2,Theorem 10.51).Therefore G is nilpotent-by-finite.

Difinition: A group G has finite Prüfer rank r=r(G) if every finitly generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this properly. Clearly subgroups and homomorphic images of groups with finite Prüfer rank also have finite Prüfer rank.

Lemma: (See [13]) If N is a maximal abelian normal dubgroup of a finite p-group G, then $r(G) \leq \frac{1}{2} r(N)(5\,r(N)+1).$

Proof: Since $C_G(N) = N,$ the factor group G/N is isomorphic with a p-group of automorphism of N. Thus G/N has perüfer rank at most $\frac{1}{2} r(N)(5r(N)-1)$ (See [15], part2, lemma 7.44), and hence $r(G) \leq \frac{1}{2} r(N)(5r(N)+1)$.

Theorem: (See [9] and [11]) If the locally soluble group G=AB with finite Prüfer rank is the product of two subgroups A and B, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B.

Proof: First, let G be a finite p-group for some prime p. If N is a maximal abelian normal subgroup of G, by Lemma 2.18 we have $r(G) \leq \frac{1}{2} r(N)(5r(N)+1).$ Hence it is enough to prove that r=r(N) is bounded by a function of the maximum s of r(A) and r(B). The socle S of N is an elementary abelian group of order p'. Clearly it is sufficient to prove the theorem for the factorizer X(S) of S. Therefore we may suppose that the group G has a triple factorization G=AB=AK=BK, where K is an elementary abelian normal subgroup of G of order p'.

Let e be the least positive integer such that A^{p^e} is contained in B. By Lemma 4.3.3 of [4], we have $|A:A\bigcap B|\le|A:A^{p^e}|\le p^{eg(s)-s^2}$ Where $g(s)=\frac{1}{2}s(3s+1)$. *Since*
$|G|=\frac{|A|.|B|}{|A\bigcap B|}=\frac{|B|.|K|}{|B\bigcap K|}$,

It follows that $|K|=|A:A\bigcap B|.|B\bigcap K|\le p^{eg(s)-s^2}p^s=p^{eg(s)-s^2+s}$. Hence $r\le eg(s)-s^2+s\le eg(s)$. Therefore it is enough to show that $e\le g(s)+3$. Therefore it is enough to show that $e\le g(s)+3$.

Clearly we may suppose that e>1. Let a be an element of A such that $a^{p^{e-1}}$ is not in B, and write $a^{p^{e-1}}$ =xb, with x in K and b in B. Then $[x,a^{p^{e-2}}]\ne 1$, because otherwise

$$b^p=(x^{-1}a^{p^{e-2}})^p=x^{-p}a^{p^{e-1}}=a^{p^{e-1}},$$

contrary to the choice of a. As K has exponent p, it follows from the usual commutator laws that .

$$[x,a^{p^{e-2}}]=\prod_{i=1}^{p^{e-2}}[x,{}_ia]^{(p^{e_i-2})}=[x,p^e.2a].$$

Thus $[K,\underbrace{G,...,G}_{\leftarrow p^{e-2}\rightarrow}]\ne 1$, and so |K|> $p^{p^{e-2}}$ since G isa finite p-group. Therefore $p^{p-2}<r\le eg(s)$. If $e\ge g(s)+4$, then $p^{e-2}\ge 2^{e-2}>(e+1)(e-4)\ge(e+1)g(s)>eg(s)$..

This contradiction shows that $e\le g(s)+3$.

Suppose now that G=AB is an arbitrary finite soluble group. For each prime p, by Corollary 2.7 there exist Sylow p-subgroups A_p of A and B_p of B such that $G_p=A_pB_p$ is a Sylow p-subgroup of G. As was shown above, r(G_p) is bounded by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by at most f(s) elements. Application of Theorem 4.2.1 of [4] yields that every subgroup of G can be generated by at most f(s)+1 elements, and hence the Prüfer rank of G is bounded by f(s)+1. This proves the theorem is the finite case.

Let G=AB be an arbitrary locally soluble group with finite Prüfer rank. If N is a finite normal subgroup of G, and X=X(N) is its factorizer, then the index $|X:A\bigcap B|$ *is finite by Lemma 1.1.5. Let Y be the core of* $A\bigcap B$ *in X*. Since the factorized group X/Y is finite, it follows from the first part of the proof that thePrüfer rank of X/Y is bounded by a function of the Prüfer ranks of A and B. As $r(N)\le r(X)\le r(Y)+r(X/Y)\le r(A)+r(X/Y)$ (e.g.see Robinson 1972, Part 1, Lemma 1.44) we obtain that there exists a function h such that $r(N)\le h(r(A),r(B))=k$, for every finite normal subgroup N of G. Clearly the same holds for every finite normal section of G.

Let T be the maximum periodic normal subgroup of G. If p is a prime, the group $\overline{T}=T/O_{p'}(T)$ is Chernikov by Lemma 3.2.5 of [4] (See also [16]). Let $\overline{J}$ be the finite residual of $\overline{T}$, and $\overline{S}$ the socle of $\overline{J}$. Since $\overline{S}$ and $\overline{T}/\overline{J}$ are finite, it follows that $r(\overline{T})\le r(\overline{J})+r(\overline{T}/\overline{J})=r(\overline{S})+r(\overline{T}/\overline{J})\le 2k$.

As the Sylow p-subgroups of T can be embedded in $\overline{T}$, they have Prüfer rank at most 2k. Application of Theorem 4.2.1 of [4] (See also [14]). yields that every finite subgroup of T can be generated by atmost 2k+1 elements. Hence r(T) $\leq 2k+1.$

The group G/T is soluble (See[15]), Part 2, Lemma 10.39), and so the setoff primes $\pi(G/T)$ is finite by Lemma 4.1.5 of [5] (See also [15]). It follows from Lemma 4.1.4 of [4] (See also [15]) that there exists in G a normal series of finite length $T \leq G_1 \leq G_2 \leq G,$ where G_1/T is torsion-free nilpotent, G_2/G_1 is torsion-free abelian, and G/G_2 is finite. Therefore

$$\begin{aligned} r(G) &\leq r(T)+r(G_1/T)+r(G_2/G_1)+r(G/G_2) \\ &\leq r(T)+r_0(G)+r(G/G_2) \\ &\leq r_0(G)+3k+1. \end{aligned}$$

By theorem 4.1.8 of [4] (See also [3]) we have that $r_0(G) \leq r_0(A)+r_0(B).$

Moreover, $r_0(A) \leq r(A)\ and\ r_0(B) \leq r(B)\ by\ Lemma\ 4.3.4$ of [4] (See also [9]). Therefore $r(G) \leq r(A)+r(B)+3k+1.$ The theorem is proved.

Lemma(See [17]**:** Every finitely generated abelian-by- polycyclic Group is residually finite.

Proof: See ([4], Lemma 4.4.1)

MAIN Theorem:

Theorem: If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Proof: Assume that G it not polycyclic-by-finite. Then G contains an abelian normal section U/V which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of U/V in G/V is also a counterexample. Hence we may suppose that G has a triple factorization G=AB=AK=BK, Where K is an abelian normal subgroup of G which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group G satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup M which is maximal with respect to the condition that G/M is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of G is polycylic-by-finite.

REFERENCE

1. Amberg, B. (1973). Factorizations of Infinite Groups. Habilitationsschrift, Universität Mainz.
2. Amberg, B. (1980). Lokal endlich-auflösbare Produkte von zwei hyperzentralen Gruppen. Archiv der Mathematik, 35(1), 228-238.
3. Amberg, B., Franciosi, S., & De Giovanni, F. (1991). Rank formulae for factorized groups. Ukrainian Mathematical Journal, 43(7-8), 1012-1018.
4. Amberg, B., Franciosi, S., & de Gioranni, F. (1992). Products of Groups. Oxford University Press Inc., New York.
5. Chernikov, N. S. (1980). Factorizations of locally finite groups. Siberian Mathematical Journal, 21(6), 890-897.
6. Amberg, B. (1985). On groups which are the product of abelian subgroups. Glasgow Mathematical Journal, 26(2), 151-156.
7. Itô, N. (1955). Über das Produkt von zwei abelschen Gruppen. Mathematische Zeitschrift, 62(1), 400-401.
8. Kegel, O. H. (1961). Produkte nilpotenter gruppen. Archiv der Mathematik, 12(1), 90-93.
9. Robinson, D. J. (1986). Soluble products of nilpotent groups. Journal of Algebra, 98(1), 183-196.
10. Wielandt, H. (1958). Über Produkte von nilpotenten Gruppen. Illinois Journal of Mathematics, 2(4B), 611-618.
11. Zaitsev, D. I. (1981). Factorizations of polycyclic groups. Mathematical Notes, 29(4), 247-252.
12. Lennox, J. C., & Roseblade, J. E. (1980). Soluble products of polycyclic groups. Mathematische Zeitschrift, 170(2), 153-154.
13. Roseblade, J.E. (1965). On groups in which every subgroup is subnormal. J. Algebra **2**, 402-412.
14. Kovács, L. G. (1968). On finite soluble groups. Mathematische Zeitschrift, 103(1), 37-39.
15. Robinson, D. J. S. (1972). Finiteness Conditions and Generalized Soluble Groups. Springer, Berlin.
16. Kegel, O. H., & Wehrfritz, B. A. F. (1973). Locally Finite Groups. North-Holland, Amsterdam.
17. Jategaonkar, A. V. (1974). Integral group rings of polycyclic-by-finite groups. Journal of Pure and Applied Algebra, 4(3), 337-343.
18. Zaitsev, D. I. (1984). Soluble factorized groups. In Structure of Groups and Subgroup Characterizations, 15-33. Akad. Nauk Ukrain. Inst. Mat. Kiev (9 in Russian).
19. Sesekin, N. F. (1968). Product of finitely-connected abelian groups. Siberian Mathematical Journal, 9(6), 1070-1072.

20. Sesekin, N. F. (1973). On the product of finitely generated Abelian groups. Mathematical notes of the Academy of Sciences of the USSR, 13(3), 266-268.
21. Cohn, P. M. (1956). A remark on the general product of two infinite cyclic groups. Archiv der Mathematik, 7(2), 94-99.
22. Rédei, L. (1950). Zur Theorie der faktorisierbaren Gruppen, I. Acta Mathematica Hungarica, 1(1), 74-98.
23. Szép, J. (1950). On factorisable, not simple groups. Acta Univ. Szeged. Sect. Sci. Math, 13, 239-241.
24. Zappa, G. (1940). Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro. In Atti Secondo Congresso Un. Mat. Ital., Bologna (pp. 119-125).
25. Neumann, B. H. (1954). Groups covered by permutable subsets. Journal of the London Mathematical Society, 1(2), 236-248.
26. Marconi, R. (1987). Sui prodotti di gruppi abeliani. Tesi di dottorato, Padova.
27. Heineken, H., & Lennox, J. C. (1983). A note on products of abelian groups. Archiv der Mathematik, 41(6), 498-501.
28. Zaitsev, D. I. (1980). Products of abelian groups. Algebra and Logic, 19(2), 94-106.
29. Huppert, B. (1967). Endliche Gruppen I. Springer, Berlin.
30. Kegel, O. H. (1965). Zur Struktur mehrfach faktorisierter endlicher Gruppen. Mathematische Zeitschrift, 87(1), 42-48.
31. Kegel, O. H. (1965). On the solvability of some factorized linear groups. Illinois Journal of Mathematics, 9(3), 535-547.
32. Franciosi, S., & de Giovanni, F. (1990). On products of locally polycyclic groups. Archiv der Mathematik, 55(5), 417-421.
33. Franciosi, S., & de Giovanni, F. (1990). On normal subgroups of factorized groups. Ricerche Mat. 39, 159-167.
34. Franciosi, S., & de Giovanni, F. (1992). On trifactorized soluble of finite rank. Geom. Dedicata, 38, 331-341.
35. Franciosi, S., & de Giovanni, F. (1992). On the Hirsch-Plotkin radical of a factorized group. Glasgow Mathematical Journal, 34(2), 193-199.
36. Franciosi, S., de Giovanni, F., Heineken, H., & Newell, M. L. (1991). On the Fitting length of a soluble product of nilpotent groups. Archiv der Mathematik, 57(4), 313-318.
37. Sysak, Y. P. (1982). Products of infinite groups. Preprint 82. 53, Akad. Nauk Ukrain. Inst. Mat. Kiev (in Russian).
38. Sysak, Y. P. (1986). Products of locally cyclic, torsion-free groups. Algebra and Logic, 25(6), 425-433.
39. Sysak, Y. P. (1988). On products of almost abelian groups. In Researches on Groups with Restrictions on Subgroups, 81-85. Akad. Nauk Ukrain. Inst. Mat. Kiev (in Russian).
40. Sysak, Y. P. (1989). Radical modules over groups of finite rank. Preprint, 89, 18.
41. Wilson, J. S. (1985). On products of soluble groups of finite rank. Commentarii Mathematici Helvetici, 60(1), 337-353.
42. Tomkinson, M. J. (1986). Products of abelian subgroups. Arch Math (Basel), 42, 107-112.