ISSN 2664-4150 (Print) & ISSN 2664-794X (Online)
South Asian Research Journal of Engineering and Technology
Abbreviated Key Title: South Asian Res J Eng Tech

| Volume-1 | Issue-1| Jun-Jul -2019 |

Research Article

Product of Periodic Groups

Behnam Razzaghmaneshi*
Assistant Professor, Department of Mathematics and Computer Science, Islamic Azad University Talesh Branch, Talesh, Iran

***Corresponding Author**
Behnam Razzaghmaneshi

Article History
Received: 06.07.2019
Accepted: 12.07.2019
Published: 30.07.2019

Abstract: A group G is called radicable if for each element x of G and for each positive integer n there exists an element y of G such that $x=y^n$. A group G is reduced if it has no non-trivial radicable subgroups. Let the hyper-((locally nilpotent) or finte) group G=AB be the product of two periodic hyper-(abelian or finite) subgroups A and B. Then the following hold: (i) G is periodic. (ii) If the Sylow p-subgroups of A and B are Chernikov (respectively: finite, trivial), then the p-component of every abelian normal section of G is Chernikov (respectively: finite, trivial).

Keywords: Residual, factorize, minimal condition, primary subgroup

2000 Mathematics subject classification: 20B32, 20D10.

INTRODUCTION

In 1968 N. F. Sesekin proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition [1]. N. F. Seskin and B.Amberg independently obtained a similar result for maximal condition around 1972 [1, 2]. Moreover, a little later the later proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the minimal condition, and its Fitting subgroup inherits the factorization. B.Amberg, N.S.Chernikov, S.Franciosi, F.de Gioranni, O.H.Kegel, J.C.Lennox, D.J.S.Robinson, J.F.Roseblade, Y.P.Sysak, J.S.Wilson and D.I.Zaitsev, development the subjects of products and factorizable group for soluble and nilpotent groups [3-12]. In 1989, S. V. Ivanov show that every counable group may be embedded in a product of two Tarski groups [13]. Now in this paper we show that if $A_1,\ldots,A_n$ is n periodic subgroups such that satisfying the minimal condition on primary subgroups, and $G=A_1\ldots A_n$, then G is also satisfies the minimal condition on primary subgroups, and $J(G)=J(A_1)\ldots J(A_n)$, where J(G) is the finite residual of group G.

Elementary Definitions and Theorems

In this section we give the elementary definition and Basic theorems whose used the proof of main theorem.

Definition

A group G is the product of two subgroups A and B if $G=AB=\{ab|a\in A,b\in B\}$. In this cas we say that G is factorized by A and B. The factorizer of S in G=AB is denoted by X(S) and defined as following: The intersection X(S) of all factorized subgroup S of G=AB containing the subgroup S is the smallest factorized subgroup of G Containing S [5].

Definition

Let P is the property pertaining to subgroups of a group G. then we define the P-radical $\rho(G)$ of G is the subgroup generated by all normal P-subgroups of G [5].

Lemma [5]

Let A,B and K be subgroups of a group G such that G=AK=BK and $A\cap K=B\cap K=1$. If K is normal in G, then $\rho(A)K=\rho(B)K$, and so this is a normal subgroup of G.

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Proof

Since $\rho(A)K/K = \rho(G/K) = \rho(B)K/K, hence\ \rho(A)K = \rho(B)K, and$ the proof is complete.

Lemma [2]

Let the group G=AB be the product of two subgroups A and B. Then:

- If A and B satisfy the maximal condition on subgroups, then G satisfies the maximal condition on normal subgroups.
- If A and B satisfy the minimal condition on subgroups, then G satifies the minimal condition on normal subgroups.

Proof

Let $(H_n)_{n\in N}$ be an ascending sequence of normal subgroups of G. Then $(A\cap H_n)_{n\in N}$ and $(B\cap H_n)_{n\in N}$ are ascending sequence of subgroups of A and B, respectively. Hence $A\cap H_n = A\cap H_{n+1}$ and $B\cap AH_n = B\cap AH_{n+1}$, for almost all n. It follows that $AH_n = AB\cap AH_n = A(B\cap AH_n) = A(B\cap AH_{n+1}) = AH_{n+1}$, and so $H_n = H_n(A\cap H_{n+1}) = AH_n \cap H_{n+1} = AH_{n+1}\cap H_{n+1} = H_{n+1}$, for almost all n. Therefore G satisfies the maximal condition on normal subgroups. The proof of (ii) is similar.

Lemma [5]

Let N be a non-periodic locally nilpotent normal subgroup of a group G such that G/N is locally finite. If x is an element of infinite order in N and n is a positive integer, the coset $x(x^n)^G$ has order exactly n in the factor group $G/(x^n)^G$. Moreover, if π is the set of prime divisors of n, the group $x^G/(x^n)^G$ is a π –group.

Proof

Let m be the least positive integer such that x^m belongs to $(x^n)^G$. Then x^m is in $(x^n)^E$ for some finitely generated subgroup E of G. Clearly $(x^n)^E$ lies in $N\cap <x^n, E>$. Since G/N is locally finite, $N\cap <x^n,E>$ has finite index in$<x^n,E>$ and so is a finitely generated nilpotent group.It follow that $(x^n)^E=<((x^m)^g)^{n/m}|g\in E>$ is a finitely generated infinite nilpotent group, which is generated by the (n/m)th powers of its elements. This is possible only if m=n. As the locally nilpotent group $x^G/(x^n)^G$ is generated by π –elements it is clearly a π –group.

Lemma

Let the group G=AB=AK=BK be the product of two subgroups A and B and a locally finite normal subgroup K. If E is a finite subgroup of A such that $\pi(E)\cap\pi(K)$ is empty, then there exists an element a of A such that E^a is contained in $A\cap B$.

Proof: See Lemma 3.2.2. of [5]

Lemma

Let π be a set of primes, and let the locally finite group G=AB=AK=BK be the product of two subgroups A and B and a normal π -subgroup K such that $A\cap K = B\cap K = 1$. If A_0 and B_0 are normal $\pi' - subgroups$ of A and B, respectively, such that $A_0 \leq B_0K$, then $N_{A_0}(A_0\cap B_0) = A_0\cap B_0$

Proof: See Lemma 3.2.3. of [5]

Lemma [5]

Let π be a set of primes, and let the locally finite group G=AB=AK=BK be the product of two subgroups A and B and a normal π -subgroup K such that $A\cap K = B\cap K = 1$. If A_0 is a normal $\pi' - sabgroup$ of A, then $B_0 = O_{\pi'}(B)\cap A_0K$ is a normal subgroup of B satisfying the following conditions.

- $A_0K = B_0K$
- ii) $N_{A_0}(A_0\cap B_0) = A_0\cap B_0$
- If A_0 is a belian or finite, then $A_0=B_0$ is a normal subgroup of G.

Proof: (i) By Lemma 2.3 we have that $O_{\pi'}(A)K = O_{\pi'}(B)K.$ Then

$$A_0K = O_{\pi'}(A)K \cap A_0K = O_{\pi'}(B)K \cap A_0K = (O_{\pi'}(B) \cap A_0K) = B_0K.$$

- This follows from (i) and lemma 2.7.
- It is clearly sufficient to show that A_0 it contained in B_0, since then $B_0 = A_0K \cap B_0 = A_0$. If A_0 is abelian, then $A_0 \cap B_0 = A_0$ by (ii), so that A_0 is contained in B_0. suppose that A_0 is finite. By lemma 2.6. there exists an element a of A such that $A_0 = A_0^a \leq A \cap B \leq N_G(B_0).$ Therefore A_0B_0 is a $\pi' - group.$ Now $A_0B_0 = A_0B_0 \cap B_0K = B_0(A_0B_0 \cap K) = B_0$, so that A_0 is contained in B_0. The lemma is proved.

Definition

Recall that a group G is called radicable if for each element x of G and for each positive integer n there exists an element y of G such that $x=y^n$. A group G is reduced if it has no non-trivial radicable subgroups.

Lemma: Let G be a locally finite group having no infinite simple sections. then:

- If G satisfies the minimal condition on p-subgroups for some prime p, then the factor group $G/O_{p'}(G)\, contains$ a Chernikov p-subgroup of finite index;
- if G satisfies the minimal condition on p-subgroups for every prime p, then the finite residual J of G is radicable abelian and the sylow subgroups of G/J are finite.

Proof [6]

Note: For see definition of locally finite group refered to" Locally Finite Groups" book of Kegel and Wehrfritz [6].

Main Theorem

Let the hyper-((locally nilpotent) or finte) group G=AB be the product of two periodic hyper-(abelian or finite) subgroups A and B. Then the following hold.

- G is periodic.
- If the Sylow p-subgroups of A and B are Chernikov (respectively: finite, trivial), then the p-component of every abelian normal section of G is Chernikov (respectively: finite, trivial).

Proof: (i) Assume that G is not periodic. Without loss of generality We may suppose that G has no non-trivial periodic normal subgroups. Then G contains a non-trivial torsion-free locally nilpotent normal subgroup K. Clearly the factorizer X (K) is also a counterexample, so that we may suppose that G has a triple factorization G=AB=AK=BK, Where $A \cap K = B \cap K = 1.$ Since A is hyper-(abelian or finite), it contains a non-trivial normal subgroup N which is either finite or an abelian p-group. As NK is normal G=AK, the intersection $NK \cap B$ is normal in B. If N is contained in B, it is a normal subgroup of $AB = G.$ This contradiction shows that N is not contained in B. Let a be an element of N\B, and write $a=b^{-1}x$, with b in B and x in K. Let q be a prime which is not in the finite set π (N).Consider the normal closures $L=x^G$ and $V=(x^q)^G$. Since x has infinite order, by Lemma.2.5 the coset $\overline{x} = xV$ has order q in $\overline{G} = G/V$, and $\overline{L} = L/V$ is a q-group. The factorizer $\overline{X}$ of $\overline{L}\ in\, \overline{G} = \overline{A}\overline{B}$ has the triple factorization

$$\overline{X} = \overline{A}*\overline{B}* = \overline{A}*\overline{L} = \overline{B}*\overline{L},$$

Where $\overline{A}* = \overline{A} \cap \overline{B}\overline{L}\ and\ \overline{B}* = \overline{B} \cap \overline{A}\overline{L}.$ By Lemma 2.8(iii) the subgroup $\overline{N} \cap \overline{A}*$ is contained in $\overline{B}*$, so that $\overline{a} = \overline{b}^{-1}\overline{x}$ is in $\overline{B}*$. Then $\overline{x} = \overline{b}\overline{a}$ belongs to $\overline{B} \cap \overline{K} = 1$. This contradiction proves.

(ii) Assume that G contains an abelian normal section M whose p-component is not a Chernikov group. Without loss of generality we may suppose that M is an abelian normal p-subgroup of G. As the factorizer X(M) is also a counterexample, it can also be assumed that G has a triple factorization G=AB=AM=BM. Since $A \cap M$ and $B \cap M$ are Cherinkov normal p-subgroups of G, we may replace G by G/($A \cap M$)($B \cap M$) and hence suppose that $A \cap M$ = $B \cap M$ =1.

By Lemma 2.3 the subgroup $O_{p'}(A)M = O_{p'}(B)M$ is normal in G. Clearly $O_{p'}$ (G) is contained in H= $O_{p'}$ (A)M. Hence $O_{p'}(G) \leq O_{p'}(H) \leq O_{p'}(A)$,so that $O_{p'}(G)$ is contained in $A \cap B$. Factoring out $O_{p'}(G)$, we may suppose that G has no non-trivial normal p' - *subgroups*, and $A \cap M$ = $B \cap M$ =1. If $O_{p'}(A)$ is not trivial, it contains a non-trivial normal subgroup A_0 of A which is either finite or abelian. Then A_0 is normal in G by Lemma 2.8(iii). This contradiction shows that $O_{p'}(A) = O_{p'}(B) = 1$ .Since A and B satisfy the minimal condition on p-subgroups, it follows from Lemma 2.10(i) that A and B contain normal p-subgroups of finite index. In particular, A and B are Cherinkov groups. The group G=AM also contains a normal p-subgroup P of finite index. Moreover, G satisfies the minimal condition on normal subgroups, by Lemma 2.4(ii). Then Also P has the minimal condition on normal subgroups and hence is a Chernikov group (Robinson 1972, Part 1, Theorem 5.21 and Corollary 2 to Theorem 5.27) [14]. Therefore G is a Cherinkov group. This contradiction completes the proof of the first statement of (ii). The proofs of the other two statements are similar proof of Theorem is complete.

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